MATHEMATICS

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UDC 519.5

A NEW SOLUTION TO POST'S PROBLEM

*ABSTRACT. In the paper there has been constructed a semirecursive recursively enumerable set B, whose Turing degree is between **0** and **1**, taking into account that* $A \leq_m B \Leftrightarrow A \leq_Q B$ *for recursively enumerable sets A.*

KEY WORDS. Post's problem, Q-reducibility.

In article [1] E. Post formulated a problem: is there a recursively enumerable set (RES) whose Turing (T-) degree will be between 0 and 1. Here 0 — T-degree of recursive sets and 1 — T-degree of T-full sets, for example, m-universal K=$\{\langle x,y\rangle: x \in W_y\}$, where $\langle x,y\rangle$ is Cantor pair number (x,y), $x,y \in \mathbb{N} = \{0,1,2,...\}$, and y — Post number RES W_y. This problem was independently solved by A. Muchnik [2] and R. Friedberg [3]. They constituted two T-comparable RES using Turing machines with oracles. It is clear that any of them solves Post's problem.

It is well known [4, p. 203] that an equivalence exists for RES A and B.

$$A \leq_T B \Leftrightarrow \overline{A} \leq_e \overline{B} \text{ where } X = N \setminus X$$

Let $\overline{B}$ be a *semirecursive* set. This means the existence of a general recursive function (GRF) f such that [5]

$$(\forall x)(D_x \subseteq \overline{B} \Leftrightarrow f(x) \in \overline{B})$$

and then e-reducibility $\overline{A}$ to $\overline{B}$ degenerates into s-reducibility, i.e.

$(\exists W - \text{RES})(x \in \overline{A} \Leftrightarrow (\exists y)(\langle x,y\rangle \in W \wedge f(y) \in \overline{B}))$

equivalent to $(\exists g - \text{GRF})(x \in \overline{A} \Leftrightarrow W_{g(x)} \cap \overline{B} \neq \emptyset)$

or $x \in A \Leftrightarrow W_{g(x)} \subseteq B \Leftrightarrow A \leq_Q B$.

Below there will be defined a GRF h such that the set

$B = \{x: (\exists y)(y > x \wedge h(y) \leq h(x))\}$

will be semirecursive [5], $K \not\leq_Q B$ and direct recalculation $b_n = \max\{x: h(x) = n\}, n \geq 0$, of set $\overline{B}$ will not be dominated by any GRF. Therefore, B is a nonrecursive set.

Let B^t denote the set

$$\{x: x \leq t \wedge (\exists y)(x < y \leq t \wedge h(y) \leq h(x))\}$$

and W_n^t denote the finite number of elements calculated in W_n to step t. Let φ_n be a one-place partial recursive function (PRF) with Kleene number n. It is necessary to meet the following requirements:

T_{2n}: $K \not\leq_Q B$ by means of PRF φ_n;

T_{2n+1}: $\overline{B}$ is not dominated by PRF φ_n,

To meet the requirement T_{2n} we will need the labels $[n], n \geq 0$. Denote by (n,t) the number with which the label $[n]$ is paired before the step t. We remark that the requirement T_{2n} in the step t *attracts attention*, if for any number $x \leq (n,t)$ the value of $\varphi_n(n,t)$ by step t is calculated and the following requirements are met:

(a) $\in K^t \Rightarrow W^t_{\varphi_n(x)} \subseteq B^t$;

(b) $x \notin K^t \Rightarrow W^t_{\varphi_n(x)} \cap \overline{B^t} \neq \emptyset$

where K^t is the finite number of elements of the set K, enumerated in K by the step t.

The requirement T_{2n+1} in the step t *attracts attention*, if for any number $x \leq (n, t+1)$ the value $\varphi_n(x)$ by step t is calculated and $\varphi_n(x)$ is larger than x-element of the direct recalculation $\overline{B^t}$.

We now turn to the determining GRF h. It will be such that $h(t+1) \leq h(t) + 1$.

STEP 0. Combining the label [0] with the number 0, we put $h(0) = 0$ and move on to the next step.

STEP t $(t \geq 1)$. We look for the requirement T_s with the smallest number s, which in step t attracts attention. If this number s does not exist, the label $[m]$ with the smallest number m, which is not yet paired with any number, should be paired with number t, we put $h(t) = m$ and move on to the next step.

Otherwise we consider the case that occurs.

A case $m = 2n$ for the adequate number n.

The label $[n]$ pair with the number t, remove labels $[k]$ for $k > m$, put $h(t) = n$ and move on to the step $t + 1$.

The case $m = 2n + 1$ for the adequate number n.

Remove labels $[k]$ for $k \geq n + 1$, pair the label $[n + 1]$ with the number t, put $h(t) = n + 1$ move on to the step $t + 1$.

Let us prove that each requirement T_m is satisfied. For that we will show that for all $m \geq 1$ lable $[m]$ *stabilizes*, i.e. in an appropriate step it will be paired with some number t_0 and will be paired with it in all the following steps. Then $K \not\leq_Q B$ by φ_m. Otherwise let n be the smallest number of the label for which it is not so. Consequently, being paired with the number t_0 in an appropriate step, it is not later removed, but is successively paired with these numbers in steps $t_0 < t_1 < t_2 <\ldots$ while $n = h(t_0) = h(t_1) = \cdots$. Therefore the set B turns out to be finite and $K \leq_Q B$ by GRF φ_n. It contradicts Q-reducibility of a nonrecursive set to a recursive one.

It will be remembered that $b_n = \max\{x: h(x) = n\}$ is n-element of the direct recalculation $\overline{B}$. If in the step t there was the case $m = 2n + 1$, then $h(t) = n + 1$. There cannot be infinitely many steps t, as for a big t it will turn out that $\varphi_n\ (n+1) < t$ and further on the requirement T_{2n+1} will not attract attention.

The author of the paper is convinced that using Q-reducibility and hypersimple sets B, for which there is a representative GRF h and which are found in any recursively enumerable T-degree [5], will allow to reprove (or obtain) some other results about such T-degrees. A question to experts: will this proof be more comprehensible?

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