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UDC 517.95+533.6

SIMULATION OF THREE-DIMENSIONAL STEADY-STATE FLOW IN THE BOTTOM OF A TROPICAL CYCLONE

ABSTRACT. The mathematically modeled three-dimensional stationary flows of an ideal polytropic gas under the action of gravity and Coriolis forces are valid for tropical cyclones of average intensity. Boundary-value problem is posed for the system of equations of gas dynamics, the solution of the problem describes the flow that occurs in smooth gas flow through the surface of a vertical cylinder of a given non-zero radius in the vicinity of an impermeable plane z = 0. It is proved that this problem is a characteristic Cauchy problem and therefore due to the analytic property of inputs data it has a unique analytic solution. The initial segment of the series that defines this analytic solution is used to construct three-dimensional steady bottom currents of natural vortices such as tropical cyclones. The coefficients of the initial segments of the series are constructed by numerically solving the systems of ordinary differential equations. The distributions of gasdynamic parameters of studied currents are presented numerically and graphically. They are consistent with field observations of tropical cyclones.

KEY WORDS. System of gas dynamics equations, the Coriolis force and gravity, spin of gas, tropical cyclone.

Formulation of boundary-value problem

The system of gas dynamics equations (SGDE) for isentropic stationary common spatial flows of polytropic gas under the action of gravity and Coriolis has the form [1, 2]

$$\begin{cases} uc_r + \frac{v}{r}c_\varphi + wc_z + \frac{(\gamma-1)}{2}c\left(u_r + \frac{u}{r} + \frac{v_\varphi}{r} + w_z\right) = 0 \\ uu_r + \frac{v}{r}u_\varphi - \frac{v^2}{r} + wu_z \frac{2}{(\gamma-1)}cc_r = av + bw\cos\varphi \\ uv_r + \frac{uv}{r} + \frac{v}{r}v_\varphi + wv_z + \frac{2}{(\gamma-1)}\frac{c}{r}c_\varphi = -au + bw\sin\varphi \\ uw_r + \frac{v}{r}w_\varphi + ww_z + \frac{2}{(\gamma-1)}cc_z = bu\cos\varphi - bv\sin\varphi - g \end{cases} \qquad (1)$$

In the system (1): x, y, z — Cartesian independent variables and in the subspace of variables x, y a polar coordinate system is introduced; (r,φ); $c = \rho^{(\gamma-1)/2}$ — gas sound velocity; γ = const > 1 — gas polytrophic index in equation of state $p = \rho^{\gamma}/\gamma$, where p and ρ denote pressure and gas density; u, v, w, respectively, denote radial, circumferential and vertical components of gas velocity vector; $a = \Omega \sin \Psi$; $b = \Omega \cos \Psi$; $\Omega = |\mathbf{\Omega}|$—modulus of the Earth's angular velocity; Ψ—latitude point O on the Earth surface, which is the beginning of a Cartesian coordinate system rotating with the Earth, if point O lies in the Northern Hemisphere, then $0 < \Psi \leq \pi/2$; and if the South, the $\pi/2 \leq \Psi < 0$, on the equator $\Psi = 0$; g = const > 0—constant free fall acceleration. In this paper, as in [1, 2], the system (1) does not consider a centrifugal acceleration: $-\mathbf{\Omega} \times (\mathbf{\Omega}(\mathbf{R}+\mathbf{r}))$, where $\mathbf{R}$—vector connecting the center of the Earth and the point O, $\mathbf{r}$—radius—vector of the current point in the coordinate system x, y, z.

For the system (1) at $z = 0$ the following initial data are set:

$$\begin{cases} c(r,\varphi,z)|_{z=0} = c_0(r,\varphi) \\ u(r,\varphi,z)|_{z=0} = u_0(r,\varphi) \\ v(r,\varphi,z)|_{z=0} = v_0(r,\varphi) \\ w(r,\varphi,z)|_{z=0} = 0 \end{cases} \qquad (2)$$

The last of the initial conditions (2) provides gas impermeability through the plane $z = 0$.

Solution of the problem (1), (2) is constructed as an infinite series

$$U(r,\varphi,z) = \sum_{k=0}^{\infty} U_k(r,\varphi)\frac{z^k}{k!}; U_k(r,\varphi) = \left.\frac{\partial^k U(r,\varphi,z)}{\partial z^k}\right|_{z=0} \qquad (3)$$

where vector U is characterized by the sought-for functions c, u, v, w as coordinates.

New conditions are added to the problem (1), (2):

$$\begin{cases} u_0 u_{or} + \dfrac{v_0}{r} u_{o\varphi} - \dfrac{2}{(\gamma-1)} c_0 c_{0r} = a v_0 \\ u_0 v_{0r} + \dfrac{u_0 v_0}{r} + \dfrac{v_0}{r} v_{0\varphi} + \dfrac{2}{(\gamma-1)} \dfrac{c_0}{r} c_{0\varphi} = -a u_0 \end{cases} \qquad (4)$$

which are necessary conditions for the resolvability of the considered characteristic Cauchy problem (CCP) (1) , (2) [1–6].

If we assume the existence of a solution to the problem (1), (2), the two relations of (4) must be provided; i.e.: c_0, u_0, v_0 cannot be arbitrary, but must satisfy the two above relations (4).

Occurrence of conditions (4) is due to the fact that the problem (1), (2) is CCP and the subspace $z = 0$ in this problem is characteristic of multiplicity of two [3, 4].

The theorem on the existence and uniqueness of analytic solutions in the standard form for CCP system of gas dynamics equations with initial data on the contact surface (i.e. the surface, through which the gas is not flowing, in this case $z = 0$) is given in [4].

Since in this paper we consider the flow with the terms specified by the influx of non-zero radius of the cylinder, the formulation of the theorem in [4] is as follows:

Theorem. *Problem (1), (2) is CCP of a standard type with data on the characteristics of multiplicity of two possessing in case of all input data analyticity the only analytical solution under the necessary conditions for CCP resolvability (4) and two additional conditions, of the type*

$$\begin{cases} u(r,\varphi,z)\big|_{r=r_{in}} = u^0(\varphi,z) < 0, \\ v(r,\varphi,z)\big|_{r=r_{in}} = v^o(\varphi,z); \qquad r_{in} = const > 0 \end{cases} \tag{5}$$

with analytic functions $u^0(\varphi, z)$, $v^0(\varphi, z)$, consistent with the conditions (2) at $z=0$, $r = r_{in}$:

$$\begin{cases} u_0(r,\varphi)\big|_{r=r_{in}} = u^0(\varphi,z)\big|_{z=0} \\ v_0(r,\varphi)\big|_{r=r_{in}} = v^0(\varphi,z)\big|_{z=0}. \end{cases} \tag{6}$$

The calculations should also consider:

$$\begin{cases} u^0(\varphi,z) \equiv u^{00} = const < 0, \\ v^o(\varphi,z) \equiv 0 \end{cases} \tag{7}$$

Construction of the first coefficients of the series

In the following works [1, 2, 5, 6] we can find the description of the construction of coefficients $c_0 = c_0(r)$; $u_0 = u_0(r)$; $v_0 = v_0(r)$, as solutions of the corresponding SODU with specific initial conditions.

Using the coefficients $\mathbf{U}_{0,}$ coefficients w_1, c_1 are uniquely determined [5, 6]. The Cauchy problems for SODU are written out, which determine $u_{11}(r)$, $u_{12}(r)$, $v_{11}(r)$, $v_{12}(r)$ where

$$u_1 = u_{11}(r)\cos\varphi + u_{12}(r)\sin\varphi,$$
$$v_1 = v_{11}(r)\cos\varphi + v_{12}(r)\sin\varphi. \tag{8}$$

w_2, c_2 are uniquely determined by these functions. Next, coefficients u_2, v_2 are constructed as

$$u_2 = u_{20}(r) + u_{21}(r)\cos\varphi + u_{22}(r)\sin\varphi + u_{23}(r)\cos 2\varphi + u_{24}(r)\sin 2\varphi;$$
$$v_2 = v_{20}(r) + v_{21}(r)\cos\varphi + v_{22}(r)\sin\varphi + v_{23}(r)\cos 2\varphi + v_{24}(r)\sin 2\varphi,$$

and w_3, c_3 are uniquely determined with their help.

As a result we get SODU of ten equations for ten sought-for functions $u_{20}(r)$, $u_{21}(r)$, $u_{22}(r)$, $u_{23}(r)$, $u_{24}(r)$, $v_{20}(r)$, $v_{21}(r)$, $v_{22}(r)$, $v_{23}(r)$, $v_{24}(r)$ (SODU specific form is given in [6]).

Once this SODU initial data (2) for an approximate description of gas flows in the vicinity of the plane $z = 0$ are solved, the following representations are used:

$$c(r,\varphi,z)\approx c_0(r)+\left[c_{10}(r)+c_{11}(r)\cos\varphi+c_{12}(r)\sin\varphi\right]\cdot z+$$
$$+\frac{(\gamma-1)}{2}\frac{1}{c_0(r)}\left[bu_1(r)\cos\varphi-bv_1(r)\cos\varphi-\frac{2}{(\gamma-1)}\times\right. \quad (9)$$
$$\left.\times\left(c_{10}(r)+c_{11}(r)\cos\varphi+c_{12}(r)\sin\varphi\right)^2\right]\cdot\frac{z^2}{2}+c_3(r,\varphi)\cdot\frac{z^3}{6};$$
$$u(r,\varphi,z)\approx u_0(r)+\left[u_{11}(r)\cos\varphi+u_{12}(r)\sin\varphi\right]\cdot z+$$
$$+\left[u_{20}+u_{21}(r)\cos\varphi+u_{22}(r)\sin\varphi+u_{23}(r)\cos 2\varphi+u_{24}(r)\sin 2\varphi\right]\cdot\frac{z^2}{2};$$
$$v(r,\varphi,z)\approx v_0(r)+\left[v_{11}(r)\cos\varphi+v_{12}(r)\sin\varphi\right]\cdot z+$$
$$+\left[v_{20}+v_{21}(r)\cos\varphi+v_{22}(r)\sin\varphi+v_{23}(r)\cos 2\varphi+v_{24}(r)\sin 2\varphi\right]\cdot\frac{z^2}{2};$$
$$w(r,\varphi,z)\approx w_2(r,\varphi)\cdot\frac{z^2}{2}+w_3(r,\varphi)\cdot\frac{z^3}{6}.$$

The presence of sum members with different degrees of z (for c and w to z^3, and for u, v at z^2) is determined by the procedure for constructing solutions of the initial boundary-value problem. Namely, in order to satisfy the system (1) at $z = 0$ the following coefficients are built c_0, u_0, v_0, $w0$, as well as $w1$, c_1. To satisfy the first differential system implications of (1) at $z = 0$, it is necessary to set up the coefficients u_1, v_1, which help to identify the coefficients w_2, c_2. In order to satisfy the second-order differential system implications of (1) at $z = 0$, it is necessary to set up the coefficients u_2, v_2, w_3, c_3. And, therefore, the use of the representation (9) provides the feasibility of system (1) up to the sum members, containing z^3.

Naturally, the construction of solutions of the Cauchy problem for the above SODU can be done numerically [5, 6].

Calculation results

In [7], the data of statistical processing of intensity indicators for tropical cyclones are offered, and primarily the velocity of air masses movement in them – according to the wind speed. Also this paper provides some general conclusions about the individual features of tropical cyclones functioning:

1. Tropical cyclones in the Atlantic and in the northwestern part of the Pacific are characterized by almost the same intensity, as well as matching the times of wind speed increase and decrease.

2. There is an approximately equal probability that any tropical cyclone can reach any intensity (any wind speed), but not more than its potential intensity.

The potential intensity stands for the upper estimate of the maximum wind speed, which is, in turn, 'is determined by the balance between the generation and dissipation of energy surface flows, most of which occur in the boundary layer of the atmosphere.'

3. These observations suggest that high intensity of tropical cyclones in the western Pacific Ocean is caused by the great length of the path of the cyclone over the warm water.

4. According to the research, the average rates of decline in the intensity of tropical cyclones over the hot and cold water are very similar. In addition, about half of reducing the intensity of a tropical cyclone falls on land.

Since tropical cyclones annually bring numerous destructions and unfortunately victims, various studies deal with this natural phenomenon for a long time putting considerable intellectual, technical and financial resources.

However, we should conclude that more detailed measurements of various parameters of individual cyclones collecting numerous statistics and attempts to reliably predict the appearance of the next tropical cyclone is not the case. And most important: understanding the causes of tropical cyclones and the definition of natural energy sources for their sustainability remains almost at the level of hypotheses not supported by any experiments, no reliable mathematic modeling.

The following formulas (9) have been used to calculate dynamic parameters and trajectories of individual particles of gas for a tropical cyclone in the following scale values of velocity and sound velocity u0 = c0 = 333 m/s and the distance r0 = 73 km. In this case, the calculations relied following dimensionless values of the input parameters: $\sin\Psi = 0.5$, $\gamma = 1.4$, $r_{in} = 1$, $c(r_{in}) = 1$, $u(r_{in}) = -0.00124$, $v(r_{in}) = 0$, $r_0 = 0.05$ [1, 2].

Figures 1 and 2 show the projection of the particle's trajectory separate gas coming from the cylinder inflow $r = r_{in}$ at a height of z = 0.001. When this swirling gas in the tropical cyclone is substantially greater than all grades tornado [2]. And moreover, in a tropical cyclone approaching the cylinder r = r0 Photo movement of the gas particles is then coming either up or down (Fig. 1).

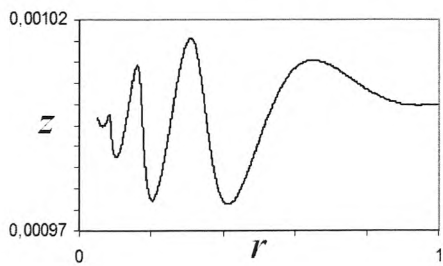

Fig. 1.

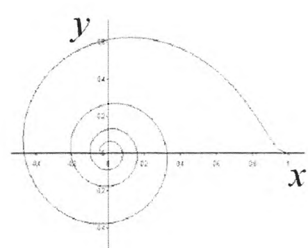

Fig. 2.

Here are the surface dynamic parameters of the tropical cyclone at a given height z = 0.001, corresponding to the dimension value of 73 m.

Figure 3 shows the surface speed of sound. This function undergoes major changes in a neighborhood undergoing Photo cylinder $r = r_0$, which corresponds to the dimension value in 3650 sq. Thus when approaching the cylinder flow values decrease the speed of sound, that is there is a vacuum gas. This vacuum is more than tornadoes all classes [2], but also not very significant. The speed of sound varies from a single value to a value of 0.997, which corresponds to the change in the density of a single value for fifteen thousandths, i.e. less than two tenths of a percent.

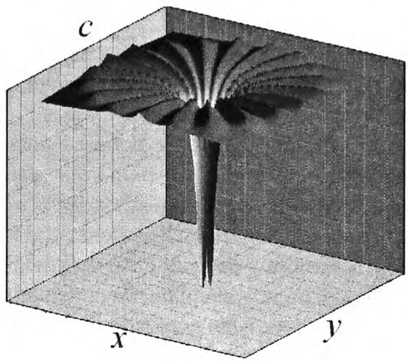

Fig. 3.

Figure 4 shows the surface of the radial velocity. The value of this velocity component gas is negative, i.e., the gas moves to the cylinder flow.

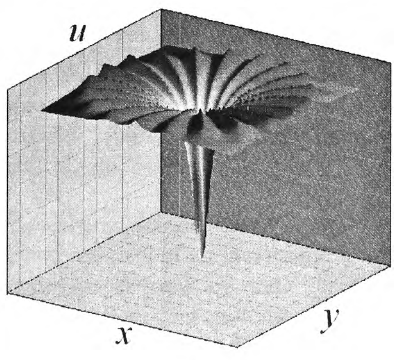

Fig. 4.

Major changes and this function in a neighborhood undergoing Photo cylinder $r = r_0$. If the cylinder inflow radial velocity values were -0.001 order, then they reach the cylinder flow values -0.025, which corresponds to the dimensional value of -8.37 m/s .

Figure 5 shows the circumferential speed of the surface. The values of this gas velocity component are positive, i.e. gas rotates in the positive direction.

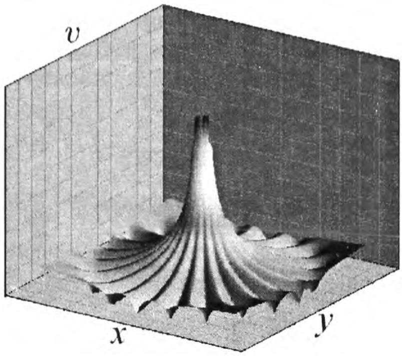

Fig. 5.

Major changes are also undergoing peripheral speed in the vicinity of the cylinder flow $r = r_0$. On the cylinder inflow peripheral speed is zero, and on the cylinder Photo its value reaches 0.17, which corresponds to the dimensional value of 52.9 m/s — storm.

And Fig. 6 shows the surface of the vertical velocity w.

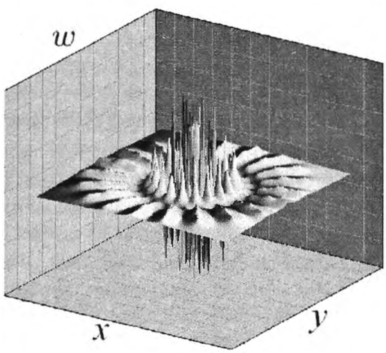

Fig. 6.

In the absolute scale the value of w is not very large, about 0.002, which corresponds to the dimension value 0.666 m/s. However, it clearly shows that when approaching the cylinder flow $r = r_0$ the value of w changes sign. That is, in my neighborhood Photo cylinder gas moves up and down. But this occurs at values of $ r\approx 0.3–0.4 $.

Summary

The research was aimed at developing a method for constructing three-dimensional steady flows, simulating the flow of air in the bottom parts of the natural upward swirling flows, such as tropical cyclones. A comparative analysis of the main gas-dynamic characteristics of air flow in the average tropical cyclone was performed. The motion trajectories of individual air particles, having local nonmonotonicities in height, were built. However, the air mass in the bottom current up to the height of 50–100 meters moves almost horizontally.

REFERENCES

1. Bautin, S.P. *Tornado i sila Koriolisa* [Tornado and the Coriolis force]. Novosibirsk: Nauka, 2008. 96 p. (in Russian).

2. Bautin, S.P., Obuhov, A.G. *Matematicheskoe modelirovanie razrushitel'nyh atmosfernyh vihrej* [Mathematical modeling of destructive atmospheric vortices]. Novosibirsk: Nauka, 2012. 152 p. (in Russian)

3. Bautin, S.P. The characteristic Cauchy problem for the quasi-linear analysis system. *Differencial'nye uravnenija — Differential equations.* 1976. Vol. 12. № 11. Pp. 2052–2063. (in Russian).

4. Bautin, S.P. *Harakteristicheskaja zadacha Koshi i ee prilozhenija v gazovoj dinamike* [The characteristic Cauchy problem and its applications in gas dynamics]. Novosibirsk: Nauka, 2009. 368 p. (in Russian).

5. Krutova, I.Ju. The problem of the motion of a gas under the action of gravity and Coriolis forces in the vicinity of an impermeable horizontal surface. *Vestnik Ural'skogo gosudarstvennogo universiteta putej soobshhenija — Ural State University of Railway Transport Bulletin.* 2012. № 1 (13). Pp. 14–21. (in Russian).

6. Krutova, I.Ju. Three-dimensional steady-state flow of gas under the action of gravity and Coriolis forces in the vicinity of an impermeable horizontal surface. *Vestnik Ural'skogo gosudarstvennogo universiteta putej soobshhenija — USURT Herald.* 2012. №3 (15). Pp. 16–23. (in Russian).

7. Emanuel, K.A. A statistical Analysis of Tropical Cyclone Intensity. *Journ. of the Atmospheric Sciences.* 2000. Vol. 128. Pp. 1139–1152.

8. Bautin, S.P., Obuhov, A.G. Mathematical modeling and numerical calculations of the flow in the bottom of a tropical cyclone. *Vestnik Tjumenskogo gosudarstvennogo universiteta — Tyumen State University Herald.* 2012. № 4. Pp. 175–182. (in Russian).

9. Obuhov, A.G. Mathematical modeling and numerical calculations of the flow in the bottom of the tornado. *Vestnik Tjumenskogo gosudarstvennogo universiteta — Tyumen State University Herald.* 2012. № 4. Pp. 183-188. (in Russian).

10. Kochin, N.E., Kibel', I.A., Roze, N.V. *Teoreticheskaja gidromehanika* [Theoretical hydromechanics]. Part 1. M.: Fizmatgiz, 1963. 403 p. (in Russian).