© B. G. AKSENOV, S. V. KARYAKINA

aksenov@tgasu.ru, karyakinaswetlan@mail.ru

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STEFAN PROBLEM AS THE LIMITING CASE OF THE PROBLEM ABOUT PHASE TRANSITION IN A TEMPERATURE RANGE

ABSTRACT. The article justifies application of a method of estimates to solve Stefan problem. The method of estimates assumes constructing differential and integral inequalities for the equations of parabolic or elliptic type that describe processes of non-stationary or stationary thermal conduction. Direct construction of such inequalities for Stefan problem is impossible because the main equation is not defined at the phase boundary.

Here Stefan problem is not viewed in its classical statement, but as a limiting case of the more general quasi-linear problem about phase transition within a range of temperatures. It is shown that under certain conditions there is an exact equality between the solution of a quasi-linear problem and some front problem. It allows using the inequalities constructed for a continuous quasi-linear problem to estimate the solution of Stefan problem. The principles which enable to construct approximate solutions of Stefan problem with various boundary conditions are formulated.

The given example is a problem about freezing-melting of moist soil. In coarse dispersed soil the pore moisture freezes (melts) at a fixed temperature. It is natural to describe this process with Stefan front problem. In finely dispersed soil the pore moisture is in the bound condition therefore the front of phase transition is not formed, and Joulean thermal is generated (absorbed) at a certain range of temperatures. For each type of finely dispersed soil the phase composition of moisture at negative temperatures is described by the so-called curve of non-frozen moisture.

Thus, the compared problems (both quasi-linear and front) are physically sensible for the process of moist soil freezing-melting.

KEYWORDS: Stefan problem, quasi-linear problem, method of estimates, moist soil freezing-melting.

The method to study, estimate and solve a non-linear thermal conduction problem with differential and integral inequalities was developed in the previous century and used to solve various applied problems [1]. Most outcomes were gained under the supervision and with the participation of Yu.S. Danielyan [2]. Application of this method to solve Stefan problem is quite peculiar due to its front structure. The solution algorithm was developed and described in [3].

The present paper is devoted to some methodological issues of applying the method of inequalities to solve Stefan problem. It describes the instruments of this method and its underlying principles, namely systematic correlation of solutions for the front problem and the problem about phase transition within a range of temperatures. This is important for further development of the method and its application to specific alternatives of the problem about phase transition, e.g. the ablation problem and many other practical issues mentioned by Yu.A. Sigunov in [4].

The given example is a problem about freezing-melting of moist soil.

In finely dispersed soils the phase transition of the pore moisture takes place within a range of temperatures. It can be explained by the fact that all the moisture is in the bound condition. Its phase composition at different temperatures is defined with the curve of non-frozen moisture w(t) where t stands for temperature. It has been experimentally proven that function w(t) does not jump and is differentiable, its derivative is a unimodal function with maximum of $t = t^*$. Function w(t) physically characterizes this material.

The mathematical description is given under A.G. Kolesnikov's model, which comes to a boundary value problem for a quasi-linear equation of thermal conduction with nonmonotonic coefficients:

$$\partial t/\partial \tau = a(\partial^2 t/\partial x^2) - (\kappa/C)\cdot(\partial w(t(x,\tau)/\partial \tau), \ \tau > 0, x > 0, \qquad (1)$$

$$t(0,\tau) = F(\tau),\ t(x,0) = t_n = \text{const},\ |t(x,\tau)| < M \qquad (2)$$

where τ — time;

x — spatial coordinate;

a — temperature conduction coefficient;

C — thermal capacity;

κ — water phase transition heat multiplied by dry unit weight;

W — non-frozen moisture content;

$F(\tau)$ — monotone constant-sign function, e.g. decreasing and negative where $F(0) \leq tn$;

M—positive constant.

The latter inequality in formulae (2) is a condition for boundedness of solution.

To make (1) simple, we assume that C and a are constant. This does not affect general conclusions as with $a = a(t)$, $C = C(t)$ Kirchhoff's substitutions let the equation have the form of (1). So we suppose that the changes have been done.

For certain grounds, if the conditions are close to thermodynamic equilibrium, the content of non-frozen water depends on the temperature solely, i.e.

$$\partial w/\partial \tau = (\partial w/\partial t)\cdot(\partial t/\partial \tau).$$

We believe that the phase transition happens at the range of temperatures $2\Delta t = t_к - t_н$ where tн—opening phase transition temperature, tк—closing phase transition temperature, tк > tн. Then with t > tк and t < tн$(\partial w/\partial t) = 0$.

Problem (1), (2) is equivalent to the following integral equation:

$$t(x,\tau)=t_0(x,\tau)+\frac{\kappa}{C}\int_0^\infty\int_0^\tau G(x,\xi,\tau-y)\frac{\partial w(t(\xi,y))}{\partial y}d\xi\,dy, \quad (3)$$

where G(x,ξ,τ-y) — Green's function for this problem; $t_0(x,\tau)$ — solution to the equation $\partial t_0/\partial\tau = a(\partial^2 t_0/\partial x^2)$, $\tau > 0$, $x > 0$ with the conditions of form (2). The expressions for t_0 and G do not have complex forms [6].

If we apply the by parts integration formula to equation (3), then after processing we get

$$t(x,\tau)=t_0(x,\tau)+V(x,\tau)+\frac{\kappa}{C}\int_0^\infty\int_0^\tau \psi(x,\xi,\tau-y)\,w(t(\xi,y))d\xi\,dy, \quad (4)$$

where

$V(x,\tau) = \kappa w(t(x,0))/C\cdot\mathrm{erf}(x/(2(a\tau)^{0,5})) - \kappa/C\cdot w(t(x,\tau))$, $\Psi(x,\xi,\tau-y) = \partial G(x,\xi,\tau-y)/\partial y$.

Stefan problem is obtained from the problems (1), (2) with $t_K = t_H = t^*$, $\Delta t = 0$. Then $\partial w(t(x,\tau))/\partial\tau = w_0\delta(\tau-\tau_n(t_*,x))$ where w_0—moisture; $\delta(\tau)$—Dirac delta function; $\tau_n(t_*,x)$—time of travel for isotherm $t = t^*$ through change point x.

If $u(x,\tau)$ is a solution for Stefan problem, then much as (1) the following problem can be written:

$$\partial u/\partial\tau = a(\partial^2 u/\partial x^2) - (\kappa/C)\cdot\delta(\tau-\tau_n(t_*,x)),\ \tau > 0,\ x > 0 \quad (5)$$

with the conditions of form (2).

The problem (1), (2) has a computational solution, e.g. using grid method. Otherwise the problem (5), (2) has a front structure. The phase transition boundary is characterized by the delta function. The thermal conduction equation describing energy transfer in the area is not defined at the phase transition boundaries. Thus, we can only give a generalized solution to this problem.

The difference approximation of differential operators alone cannot lead to a computational solution. What we need is additional methods of transition front calculation. There are plenty of such methods [4], but they are usually effective with a single transition front. If there are multiple transition fronts, and they can emerge and merge both together and with the area boundaries, the problem becomes more complicated as it is necessary to trace the evolution of each front in particular. It is very helpful to apply the shock-capturing methods based on "smearing" the front within some range of temperatures. This results in a thermal conduction non-linear equation. However the solution in a form of the temperature range does not allow locating the front. Isotherm $t(x,\tau) = t_*$ under this statement of the problem is not the surface where the transition latent heat is generated.

Similarly, (3), (4), for Stefan problem we put:

$$u(x,\tau)=t_0(x,\tau)+\frac{\kappa w_0}{C}\int_0^\infty G(x,\xi,\tau-\tau_n(t_*,x))d\xi, \quad (6)$$

$$u(x,\tau)=t_0(x,\tau)+V_1(x,\tau)+\frac{\kappa}{C}\int_0^{\infty}\int_0^{\tau}\psi(x,\xi,\tau-y)w_c(u(\xi,y))d\xi dy, \quad (7)$$

where

$V_1(x,\tau) = \kappa w_c(u(\xi,\tau))/C\cdot erf(x/(2(a\tau)^{0,5}) - \kappa/C\cdot w_c(u(x,\tau))$,
$w_c(u) = w_0$ if $u \geq t_*$ and $w_c(u) = 0$ if $u < t_*$.

The mean value theorems allow changing equation (3) into the following form

$$t(x,\tau)=t_0(x,\tau)+\frac{\kappa w_0}{C}\int_0^{\infty}G(x,\xi,\tau-\tau_n(t_*,\xi))d\xi, \quad (8)$$

where $t_c \in (t_н, t_к)$ with t_c being undefined in relation to t*. Equation (5), can be changed into the following form

$$u(x,\tau)=t_0(x,\tau)+\frac{\kappa w_0}{C}\int_0^{\infty}\int_0^{\tau}G(x,\xi,\tau-y)\frac{\partial w(u(\xi,y)+u_c)}{\partial y}d\xi dy, \quad (9)$$

where $-\Delta t \leq u_c \leq \Delta t$.

Changes (8), (9) under the thermal conduction theory mean that moving radiants distributed within some range are equivalent to a single radiant of a capacity equal to the cumulative capacity of the former, and vice versa—a single radiant can be replaced with the equivalent distributed ones if their total capacity allows it. This relation is very important for our method and we will formulate it in more details.

Principle 1. To solve the problem of thermal conduction with distributed radiants there can be taken a single radiant of a capacity equal to the distributed radiants cumulative capacity, leading to the formation of the similar temperature field [6].

For example, if $t(x,\tau)$ solves problem (1), (2), then we can take function $t_*(x,\tau)$, $t_н \leq t_* \leq t_к$ allowing equation t = u, if u solves the problem (5), (2).

Otherwise, for a problem in the form of (5), (2) we can take such a problem in the form of (1), (2) which will allow equation t = u, including cases with t* = const.

Although the direct change (from t to u) can have a single value, the reverse change can have an endless number of solutions. For example, one can change interval [tн; tк] and the form of curve w(t). The following equation is the necessary criterion for an equivalent replacement:

$$\int_{t_н}^{t_к}\frac{\partial w}{\partial \tau}dx = w_0. \quad (10)$$

It should be mentioned that the front problem solved by such change is not always a classical Stefan problem where t* = const.

Let us formulate another important principle.

Principle 2. If there are heat exchange process with the phase transition front in the areas of a similar geometry, and their initial and boundary conditions are equal, and the environmental features are equal, but the phase transition temperature t* in one of them is higher than in the other: $t_{*1} \geq t_{*2}$ then if the process is monotonic in the whole area $t_1 \geq t_2$ or $t_1 \leq t_2$ with the inequality sign showing the direction of the phase transition.

To give an example of the above, let us quote the following problems:

$$\partial t_{i1}/\partial \tau = a(\partial^2 t_{i1}/\partial x^2),\ 0 < x < \xi_i, \tag{11}$$

$$\partial t_{i2}/\partial \tau = a(\partial^2 t_{i2}/\partial x^2),\ \xi_I < x < \infty, \tag{12}$$

$$t_{i1}(0,\tau) = t_c,\ t_i(x,0) = t_0 = \text{const}. \tag{13}$$

The conditions at the interphase boundary $x = \xi_i$:

$$t_{11} = t_{12} = t_{1*} = \text{const},\ t_{21} = t_{22} = t_{2*}(x,\tau) \tag{14}$$

where ti*—crystallization (melting) temperature,

$$\lambda_1(\partial t_{i1}/\partial x\,|_{x=\xi i}) = \lambda_2(\partial t_{i2}/\partial x\,|_{x=\xi i}) = \kappa w_0(d\xi_i/d\tau) \tag{15}$$

To be certain, let us assume $t_0 > t1^*$, $t_0 > t2^*$, $t_c < t_1^*$, $t_c < t_2^*$, t_c—monotonic decreasing function.

With i = 1 we have a classical Stefan problem. If with any x, τ the condition is $t_{1*} \leq t_{2*}(x,\tau)$ then it is evident that $\xi_1 \geq \xi_2$ and using the comparison theorems [7], it is possible to show that

$$t_1 \leq t_2. \tag{16}$$

Otherwise, if $t_{1*} \geq t_{2*}(x,\tau)$ then $\xi_1 \leq \xi_2$ and

$$t_1 \geq t_2. \tag{17}$$

Here $t_i = t_{i1}$ if $0 < x < \xi_i$ and $t_i = t_{i2}$ if $\xi_I < x < \infty$.

We believe that the systematic use of principles 1 and 2 for an approximate solution of Stefan problem contributes to further study of this problem. It helps to compare the accurate development of Stefan problem and the solution of the problem with the "smeared" front. The latter can give the upper-bound and lower-bound estimates for the front and temperature co-ordinate.

The above described principles can be applied only if differential and integral inequalities for the continual and front problems are used systematically. This article does not allow any detailed description of them. Differential inequalities (comparison theorems) have been formulated by Fridman [7], and integral inequalities are given in [1].

The following example serves as demonstration. The properties of function w(t) allow making another change to equation (1)

$$\partial t/\partial \tau = \lambda/(C + \varphi(t))\cdot(\partial^2 t/\partial x^2),\ \varphi(t) = \kappa(dw/dt),\ \tau > 0,\ x > 0, \tag{18}$$

(λ—thermal conduction coefficient) and to equation (5)

$$\partial u/\partial \tau = \lambda/(C + \kappa w_0 \cdot \delta(u-t_*) \cdot (\partial^2 u/\partial x^2), \tau > 0, x > 0. \quad (19)$$

Suppose that in problem (18) $2\Delta t = t_к - t_н$—the front 'smearing' zone in a shock-capturing problem, $t^* = (tк + tн)/2$.

Taking into account (9) and (18) Stefan problem can be put in the form of:

$$\partial u/\partial \tau = \lambda/(C + \varphi(u + u_c)) \cdot (\partial^2 u/\partial x^2), \tau > 0, x > 0, \quad (20)$$

$$u(0,\tau) = F(\tau), u(x,0) = t_n = \text{const}, \ |u(x,\tau)| < M. \quad (21)$$

u_c is unknown, but estimates $u(x,\tau)$ can be gained. To gain the estimates of function $u(x,\tau)$ let us solve two problems:

$$\partial y_i/\partial \tau = \lambda/(C + \varphi(y_i + (-1)^i \Delta t)) \cdot (\partial^2 y_i/\partial x^2), \tau > 0, x > 0, i = 1.2 \quad (22)$$

with the conditions in the form of (21). These problems can have computational solutions.

If S is a co-ordinate of isotherm $u = t^*$, and σ_1 and σ_2—of isotherms $y_1 = t_* - \Delta t$ and $y_2 = t_* + \Delta t$ the process monotony and principle 2 lead to the conclusion that

$$\sigma_1 \leq S \leq \sigma_2. \quad (23)$$

Thus, to find the accuracy value under the conditions of the front 'smearing', it is necessary to solve two problems in the form of (1), (2).

Conclusion

Paper [3] shows that the above mentioned technique makes it possible to solve Stefan problem in the form of recurrent procedure with the proven convergence. Moreover, the condition of monotonic $F(\tau)$ function can be revoked.

The solution can be found for any $F(\tau)$ function with limited options.

Papers [8–10] describe the way of adapting the inequalities method to the solution of the problems with axiality and centrosymmetry. In practice it is used to calculate the temperature fields of pipelines buried in a moist ground.

The inequalities method is being applied to solve 2D-problems. A problem of steady thermal conduction (elliptic equation) has been solved. A problem of 2D-area has not been solved yet. This lies in a perspective.

The next objective is to solve the problem of ablation [4].

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