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NONLINEAR SIMULATION OF SURFACE WAVES IN A TWO-PHASE MIXTURE

ABSTRACT. The paper discusses a nonlinear boundary value problem of wave propagation on the free surface of a two-phase (disperse) mixture. To solve the problem, a method of time-variant frequency is applied, which was devised for the solution of nonlinear wave problems with dissipation. A solution has been found to a third order approximation using a small-amplitude parameter. Nonlinear wave frequency and near-surface Stokes flow rate have been defined. They are time functions depending on the wave length and elevation (as in the classical case) and on the dispersed phase concentration. Nonlinear paths of carrier and dispersed phase particles have been found. Basing on the obtained expression for the dispersed particles paths, the nonlinear effect has been described: in wave motion dispersed particles that are heavier than carrier phase particles float, while lighter particles sink.

KEY WORDS. Nonlinear surface waves, two-phase medium.

An infinitely deep layer of two-phase mixture, bounded from above by free surface $z^* = \xi^* (t^*,x^*)$ is considered. It is assumed: the carrier phase is ideal incompressible liquid, viscosity of which only occurs at the interface; the disperse phase is spherical particles of a radius; a wave with λ (λ >> ξ * max, λ >> a) length propagates in the x^*; $k/\xi^*_{max} = \varepsilon$ direction—small wave parameter, $k = 2\pi/\lambda$—wave number. Physical (dimensional) values, where necessary, are marked with asterisks. Setting wave frequency (phase velocity) an unknown time function, we can write the nonlinear boundary value problem [4] in a dimensionless form

$$-\frac{\alpha_o}{(t/c)'}\frac{\partial\gamma}{\partial t}+(1-\alpha_o)\nabla\cdot \mathrm{v}_1-\varepsilon\alpha_o\nabla\cdot(\gamma \mathrm{v}_1)=0,\quad \frac{1}{(t/c)'}\frac{\partial\gamma}{\partial t}+\nabla\cdot \mathrm{v}_2+\varepsilon\nabla\cdot(\gamma \mathrm{v}_2)=0,\ (1)$$

$$\left(\mu_1+\frac{s}{2}\mu_1\alpha_o\right)\frac{1}{(t/c)'}\frac{\partial \mathrm{v}_1}{\partial t}-\frac{s\mu_1\alpha_o}{2(t/c)'}\frac{\partial \mathrm{v}_2}{\partial t}-r\alpha_o(\mathrm{v}_2-\mathrm{v}_1)+\nabla p+\varepsilon\left[\mu_1\left(1+\frac{s}{2}\alpha_0\right)(\mathrm{v}_1\nabla)\mathrm{v}_1-\right.$$

$$\left.-\frac{s}{2}\mu_1\alpha_0(\mathrm{v}_2\nabla)\mathrm{v}_2-r\alpha_0\gamma(\mathrm{v}_2-\mathrm{v}_1)-\frac{s\mu_1\alpha_o\gamma}{2(t/c)'}\left(\frac{\partial \mathrm{v}_2}{\partial t}-\frac{\partial \mathrm{v}_1}{\partial t}\right)\right]+\varepsilon^2\frac{s}{2}\mu_1\alpha_o\gamma\left[(\mathrm{v}_1\nabla)\mathrm{v}_1-\right.$$

$$\left.-(\mathrm{v}_2\nabla)\mathrm{v}_2\right]=0,$$

$$\left[\mu_2+\frac{s}{2}\mu_1(1-\alpha_o)\right]\frac{1}{(t/c)'}\frac{\partial v_2}{\partial t}-\frac{s\mu_1(1-\alpha_o)}{2(t/c)'}\frac{\partial v_1}{\partial t}+r(1-\alpha_o)(v_2-v_1)+\nabla p+\varepsilon\left\{\left[\mu_2+\right.\right.$$

$$\left.\left.+\frac{s}{2}\mu_1(1-\alpha_0)\right](v_2\nabla)v_2-\frac{s}{2}\mu_1(1-\alpha_0)(v_1\nabla)v_1-r\alpha_0\gamma(v_2-v_1)-\frac{s\mu_1\alpha_o\gamma}{2(t/c)'}\left(\frac{\partial v_2}{\partial t}-\frac{\partial v_1}{\partial t}\right)\right\}+$$

$$+\varepsilon^2\frac{s}{2}\mu_1\alpha_o\gamma\left[(v_1\nabla)v_1-(v_2\nabla)v_2\right]=0,\quad v_1=(v_{1x},v_{1z}),\quad v_2=(v_{2x},v_{2z}),$$

$$\frac{1}{(t/c)'}\frac{\partial\xi}{\partial t}-(1-\alpha_o)v_{1z}-\alpha_o v_{2z}+\varepsilon\left\{\frac{\partial\xi}{\partial x}\left[(1-\alpha_o)v_{1x}+\alpha_o v_{2x}\right]-\alpha_o\gamma(v_{2z}-v_{1z})\right\}+$$

$$+\varepsilon^2\alpha_o\gamma\frac{\partial\xi}{\partial x}(v_{2x}-v_{1x})=0,\quad z=\varepsilon\xi(t,x),$$

$$p-v_o^2\xi+\varepsilon\alpha_o(\mu_1-\mu_2)v_o^2\gamma\xi=0,\quad v_o^2=g/kc_o^2,\quad z=\varepsilon\xi(t,x),$$

$$v_{iz}\to 0,\quad v_{ix}\to 0,\quad z\to-\infty,\quad i=1,2.$$

Dimensionless quantities are associated with physical equations

$t^*=kc^*t^*$, $x=kx^*$, $z=kz^*$, $\mu_i=\rho_i^0/\rho^0$, $r=R/\rho^0kc_0$, $\rho^0=(1-\alpha_0)\rho_1^0+\alpha_0\rho_2^0$,

$c=c^*(t)/c_0=\omega^*(t)/\omega_0$, $\alpha^*=\varepsilon\alpha_0\gamma$, $k\xi^*=\varepsilon\xi$, $\mathbf{v}_i^*=\varepsilon c_0\mathbf{v}_i$, $p^*=P_i-P_a+\rho_i^0gz^*=\varepsilon\rho^0c_0^2p$.

Here ω_0, c_0 is frequency, phase velocity of a linear wave; α_0, α^* is disperse phase concentration, its disturbance; $\mathbf{v}_i^*, P_i, \rho_i^0$ is velocity vector, pressure, true density of the i-th phase; ρ^0, P_a is quiescent mixture density, atmospheric pressure; $R=9\eta/a^2$ is interphase friction coefficient (η—dynamic viscosity), coefficient $s=1; 0$ depending on account of the added mass force.

Maclaurin expansion of the boundary conditions are reduced to the surface $z=0$. Unknown functions are defined as ε series:

$$v_i=\sum_{n=1}^{\infty}\varepsilon^{n-1}v_{in},\quad p=\sum_{n=1}^{\infty}\varepsilon^{n-1}p_n,\quad \xi=\sum_{n=1}^{\infty}\varepsilon^{n-1}\xi_n,\quad \gamma=\sum_{n=1}^{\infty}\varepsilon^{n-1}\gamma_n,\quad c=1+\sum_{n=1}^{\infty}\varepsilon^n a_n(t).$$

(2) Substituting (2) series in the equations and boundary conditions (laid out in $z=0$ neighborhood) (1) and equating coefficients of ε^0, ε^1, ε^2, we obtain an asymptotic representation of (1) problem in the first three approximations. In the first approximation, the problem takes the form of: when ε^0

$$-\alpha_o\frac{\partial\gamma_1}{\partial t}+(1-\alpha_o)\nabla\cdot v_{11}=0,\quad \frac{\partial\gamma_1}{\partial t}+\nabla\cdot v_{21}=0, \tag{3}$$

$$\left(\mu_1+\frac{s}{2}\mu_1\alpha_o\right)\frac{\partial v_{11}}{\partial t}-\frac{s}{2}\mu_1\alpha_o\frac{\partial v_{21}}{\partial t}-r\alpha_o(v_{21}-v_{11})+\nabla p_1=0,$$

$$\left(\mu_2+\frac{s}{2}\mu_1(1-\alpha_o)\right)\frac{\partial v_{21}}{\partial t}-\frac{s}{2}\mu_1(1-\alpha_o)\frac{\partial v_{11}}{\partial t}+r(1-\alpha_o)(v_{21}-v_{11})+\nabla p_1=0,$$

$$\mathrm{v}_{11}=\left(\mathrm{v}_{1x1},\mathrm{v}_{1z1}\right),\quad \mathrm{v}_{21}=\left(\mathrm{v}_{2x1},\mathrm{v}_{2z1}\right),$$

$$\frac{\partial\xi_1}{\partial t}=\left(1-\alpha_o\right)\mathrm{v}_{1z1}+\alpha_o\mathrm{v}_{2z1},\quad z=0,$$

$$p_1-\nu_o^2\xi_1=0,\quad z=0;$$

for the second approximation: at ε^1

$$-\alpha_o\frac{\partial\gamma_2}{\partial t}+\left(1-\alpha_o\right)\nabla\cdot\mathrm{v}_{12}=\alpha_0\nabla\cdot\left(\gamma_1\mathrm{v}_{11}\right),\quad \frac{\partial\gamma_2}{\partial t}+\nabla\cdot\mathrm{v}_{22}=-\nabla\cdot\left(\gamma_1\mathrm{v}_{21}\right),\tag{4}$$

$$\left(\mu_1+\frac{s}{2}\mu_1\alpha_o\right)\frac{\partial\mathrm{v}_{12}}{\partial t}-\frac{s}{2}\mu_1\alpha_o\frac{\partial\mathrm{v}_{22}}{\partial t}-r\alpha_o\left(\mathrm{v}_{22}-\mathrm{v}_{12}\right)+\nabla p_2=r\alpha_o\gamma_1\left(\mathrm{v}_{21}-\mathrm{v}_{11}\right)-$$

$$-\left(\mu_1+\frac{s}{2}\mu_1\alpha_o\right)\left(\mathrm{v}_{11}\nabla\right)\mathrm{v}_{11}+\frac{s}{2}\mu_1\alpha_o\left(\mathrm{v}_{21}\nabla\right)\mathrm{v}_{21}+\frac{s}{2}\mu_1\alpha_o\gamma_1\left(\frac{\partial\mathrm{v}_{21}}{\partial t}-\frac{\partial\mathrm{v}_{11}}{\partial t}\right),$$

$$\left(\mu_2+\frac{s}{2}\mu_1\left(1-\alpha_o\right)\right)\frac{\partial\mathrm{v}_{22}}{\partial t}-\frac{s}{2}\mu_1\left(1-\alpha_o\right)\frac{\partial\mathrm{v}_{12}}{\partial t}+\nabla p_2+r\left(1-\alpha_o\right)\left(\mathrm{v}_{22}-\mathrm{v}_{12}\right)=$$

$$=-\left(\mu_2+\frac{s}{2}\mu_1\left(1-\alpha_o\right)\right)\left(\mathrm{v}_{21}\nabla\right)\mathrm{v}_{21}+\frac{s}{2}\mu_1\left(1-\alpha_o\right)\left(\mathrm{v}_{11}\nabla\right)\mathrm{v}_{11}+r\alpha_o\gamma_1\left(\mathrm{v}_{21}-\mathrm{v}_{11}\right)+$$

$$+\frac{s}{2}\mu_1\alpha_o\gamma_1\left(\frac{\partial\mathrm{v}_{21}}{\partial t}-\frac{\partial\mathrm{v}_{11}}{\partial t}\right),\qquad \mathrm{v}_{12}=\left(\mathrm{v}_{1x2},\mathrm{v}_{1z2}\right),\quad \mathrm{v}_{22}=\left(\mathrm{v}_{2x2},\mathrm{v}_{2z2}\right),$$

$$\frac{\partial\xi_2}{\partial t}-\left(1-\alpha_o\right)\mathrm{v}_{1z2}-\alpha_o\mathrm{v}_{2z2}=\alpha_o\gamma_1\left(\mathrm{v}_{2z1}-\mathrm{v}_{1z1}\right)-\frac{\partial\xi_1}{\partial x}\left[\left(1-\alpha_o\right)\mathrm{v}_{1x1}+\alpha_o\mathrm{v}_{2x1}\right]+$$

$$+\left(1-\alpha_o\right)\xi_1\frac{\partial\mathrm{v}_{1z1}}{\partial z}+\alpha_o\xi_1\frac{\partial\mathrm{v}_{2z1}}{\partial z},\quad z=0,$$

$$p_2-\nu_o^2\xi_2=\alpha_o\left(\mu_2-\mu_1\right)\nu_o^2\gamma_1\xi_1-\xi_1\frac{\partial p_1}{\partial z},\quad z=0;$$

for the third approximation: at ε^2

$$-\alpha_o\frac{\partial\gamma_3}{\partial t}+\left(1-\alpha_o\right)\nabla\cdot\mathrm{v}_{13}=\alpha_0\nabla\cdot\left(\gamma_1\mathrm{v}_{12}+\gamma_2\mathrm{v}_{11}\right)+\alpha_0\frac{\partial\gamma_1}{\partial t}\left(ta_2\right)',\tag{5}$$

$$\frac{\partial\gamma_3}{\partial t}+\nabla\cdot\mathrm{v}_{23}=-\nabla\cdot\left(\gamma_1\mathrm{v}_{22}+\gamma_2\mathrm{v}_{21}\right)-\frac{\partial\gamma_1}{\partial t}\left(ta_2\right)',$$

$$\left(\mu_1+\frac{s}{2}\mu_1\alpha_o\right)\frac{\partial\mathrm{v}_{13}}{\partial t}-\frac{s}{2}\mu_1\alpha_o\frac{\partial\mathrm{v}_{23}}{\partial t}-r\alpha_o\left(\mathrm{v}_{23}-\mathrm{v}_{13}\right)+\nabla p_3=r\alpha_o\gamma_1\left(\mathrm{v}_{22}-\mathrm{v}_{12}\right)-$$

$$-\left(\mu_1+\frac{s}{2}\mu_1\alpha_o\right)\left[\left(\mathrm{v}_{11}\nabla\right)\mathrm{v}_{12}+\left(\mathrm{v}_{12}\nabla\right)\mathrm{v}_{11}\right]+\frac{s}{2}\mu_1\alpha_o\left[\left(\mathrm{v}_{21}\nabla\right)\mathrm{v}_{22}+\left(\mathrm{v}_{22}\nabla\right)\mathrm{v}_{21}\right]+$$

$$+r\alpha_o\gamma_2(\mathrm{v}_{21}-\mathrm{v}_{11})+\frac{s}{2}\mu_1\alpha_o\gamma_1\left(\frac{\partial \mathrm{v}_{22}}{\partial t}-\frac{\partial \mathrm{v}_{12}}{\partial t}\right)+\frac{s}{2}\mu_1\alpha_o\gamma_2\left(\frac{\partial \mathrm{v}_{21}}{\partial t}-\frac{\partial \mathrm{v}_{11}}{\partial t}\right)+$$

$$+\mu_1\left[\frac{s}{2}\alpha_o\frac{\partial \mathrm{v}_{21}}{\partial t}-\left(1+\frac{s}{2}\alpha_o\right)\frac{\partial \mathrm{v}_{11}}{\partial t}\right](ta_2)'-\frac{s}{2}\mu_1\alpha_o\gamma_1\left[(\mathrm{v}_{11}\nabla)\mathrm{v}_{11}-(\mathrm{v}_{21}\nabla)\mathrm{v}_{21}\right],$$

$$\left(\mu_2+\frac{s}{2}\mu_1(1-\alpha_o)\right)\frac{\partial \mathrm{v}_{23}}{\partial t}-\frac{s}{2}\mu_1(1-\alpha_o)\frac{\partial \mathrm{v}_{13}}{\partial t}+\nabla p_3+r(1-\alpha_o)(\mathrm{v}_{23}-\mathrm{v}_{13})=$$

$$=-\left(\mu_2+\frac{s}{2}\mu_1(1-\alpha_o)\right)\left[(\mathrm{v}_{21}\nabla)\mathrm{v}_{22}+(\mathrm{v}_{22}\nabla)\mathrm{v}_{21}\right]+\frac{s}{2}\mu_1\alpha_o\gamma_1\left(\frac{\partial \mathrm{v}_{22}}{\partial t}-\frac{\partial \mathrm{v}_{12}}{\partial t}\right)+$$

$$+\frac{s}{2}\mu_1(1-\alpha_o)\left[(\mathrm{v}_{11}\nabla)\mathrm{v}_{12}+(\mathrm{v}_{12}\nabla)\mathrm{v}_{11}\right]+r\alpha_o\gamma_1(\mathrm{v}_{22}-\mathrm{v}_{12})+r\alpha_o\gamma_2(\mathrm{v}_{21}-\mathrm{v}_{11})+$$

$$+\frac{s}{2}\mu_1\alpha_o\gamma_2\left(\frac{\partial \mathrm{v}_{21}}{\partial t}-\frac{\partial \mathrm{v}_{11}}{\partial t}\right)+\left[\left(\mu_2+\frac{s}{2}\mu_1(1-\alpha_o)\right)\frac{\partial \mathrm{v}_{21}}{\partial t}+\frac{s}{2}\mu_1(1-\alpha_o)\frac{\partial \mathrm{v}_{11}}{\partial t}\right](ta_2)'-$$

$$-\frac{s}{2}\mu_1\alpha_o\gamma_1\left[(\mathrm{v}_{11}\nabla)\mathrm{v}_{11}-(\mathrm{v}_{21}\nabla)\mathrm{v}_{21}\right],\quad \mathrm{v}_{13}=(\mathrm{v}_{1x3},\mathrm{v}_{1z3}),\quad \mathrm{v}_{23}=(\mathrm{v}_{2x3},\mathrm{v}_{2z3}),$$

$$\frac{\partial\xi_3}{\partial t}-(1-\alpha_o)\mathrm{v}_{1z3}-\alpha_o\mathrm{v}_{2z3}=\alpha_o\gamma_1(\mathrm{v}_{2z2}-\mathrm{v}_{1z2})+\alpha_o\gamma_2(\mathrm{v}_{2z1}-\mathrm{v}_{1z1})-\frac{\partial\xi_1}{\partial t}(ta_2)'-$$

$$-\frac{\partial\xi_1}{\partial x}\left[(1-\alpha_o)\mathrm{v}_{1x2}+\alpha_o\mathrm{v}_{2x2}\right]-\frac{\partial\xi_2}{\partial x}\left[(1-\alpha_o)\mathrm{v}_{1x1}+\alpha_o\mathrm{v}_{2x1}\right]+(1-\alpha_o)\xi_1\frac{\partial \mathrm{v}_{1z2}}{\partial z}+$$

$$+\alpha_o\xi_1\frac{\partial \mathrm{v}_{2z2}}{\partial z}+(1-\alpha_o)\xi_2\frac{\partial \mathrm{v}_{1z1}}{\partial z}+\alpha_o\xi_2\frac{\partial \mathrm{v}_{2z1}}{\partial z}+\frac{1}{2}(1-\alpha_o)\xi_1^2\frac{\partial^2 \mathrm{v}_{1z1}}{\partial z^2}+\frac{1}{2}\alpha_o\xi_1^2\frac{\partial^2 \mathrm{v}_{2z1}}{\partial z^2}$$

$$-\xi_1\frac{\partial\xi_1}{\partial x}\left[(1-\alpha_o)\frac{\partial \mathrm{v}_{1x1}}{\partial z}+\alpha_o\frac{\partial \mathrm{v}_{2x1}}{\partial z}\right]+\alpha_0\gamma_1\xi_1\left(\frac{\partial \mathrm{v}_{2z1}}{\partial z}-\frac{\partial \mathrm{v}_{1z1}}{\partial z}\right)+\alpha_0\frac{\partial\gamma_1}{\partial z}\xi_1(\mathrm{v}_{2z1}-\mathrm{v}_{1z1})-$$

$$-\alpha_0\gamma_1\frac{\partial\xi_1}{\partial x}(\mathrm{v}_{2x1}-\mathrm{v}_{1x1}),\quad z=0,$$

$$p_3-v_o^2\xi_3=\alpha_o(\mu_2-\mu_1)v_o^2\left(\gamma_1\xi_2+\gamma_2\xi_1+\xi_1^2\frac{\partial\gamma_1}{\partial z}\right)-\xi_1\frac{\partial p_2}{\partial z}-\xi_2\frac{\partial p_1}{\partial z}-\frac{1}{2}\xi_1^2\frac{\partial^2 p_1}{\partial z^2},\quad z=0.$$

Linear problem (3) solution has [4] the form:

$$\mathrm{v}_{ix1}=Ae^{z-bt}(M_i\cos\chi+N_i\sin\chi),\quad \mathrm{v}_{iz1}=Ae^{z-bt}(M_i\sin\chi-N_i\cos\chi),\quad i=1,2,$$

$$\gamma_1=0,\quad p_1=Ae^{z-bt}\cos\chi,\quad \xi_1=Av_o^{-2}e^{-bt}\cos\chi,\quad \chi=x-t+\theta,$$

where $M_1=\frac{1}{b^2+1}\left[1+2(b^2+1)\mu_1\mu_2(1-\mu_1)(s+2\mu_2)/d\right]$,

$$M_2=\frac{1}{b^2+1}\left[1+2(b^2+1)\mu_1^2(1-\mu_2)(s+2\mu_2)/d\right],$$

$$N_1 = \frac{1}{b^2+1}\left[-b + 2\left(b^2+1\right)\mu_2\left(1-\mu_1\right)\left(2r - b\mu_1\left(s+2\mu_2\right)\right)/d\right],$$

$$N_2 = \frac{1}{b^2+1}\left[-b + 2\left(b^2+1\right)\mu_1\left(1-\mu_2\right)\left(2r - b\mu_1\left(s+2\mu_2\right)\right)/d\right],$$

$$d = \left(2r - b\mu_1\left(s+2\mu_2\right)\right)^2 + \mu_1^2\left(s+2\mu_2\right)^2.$$

Here b is dimensionless damping decrement, $\beta = bkc_0$ is dimensional decrement. For damping decrement and phase velocity in [4] the expressions were obtained $c_o^2 = c_g^2 + c_d^2 + c_r^2$,

$$\beta = k\left\{\left[-\frac{\delta}{2} + \sqrt{\frac{\delta^2}{4} + \frac{\psi^3}{27}}\right]^{\frac{1}{3}} + \left[-\frac{\delta}{2} - \sqrt{\frac{\delta^2}{4} + \frac{\psi^3}{27}}\right]^{\frac{1}{3}} + \frac{2R/\rho^o k}{3\mu_1\left(s+2\mu_2\right)}\right\},$$

where

$$c_g^2 = \frac{g}{k},\quad c_d^2 = \frac{2\alpha_o\left(1-\alpha_o\right)\left(\rho_1^o - \rho_2^o\right)^2}{\rho_1^o\left(s\rho^o + 2\rho_2^o\right)}\,\frac{g}{k},\quad c_r^2 = \frac{\beta}{k^2}\left(3\beta - \frac{4R\rho^o}{\rho_1^o\left(s\rho^o + 2\rho_2^o\right)}\right),$$

$$\psi = \frac{3\mu_1\left(s+2\mu_2\right)\left(s\mu_1 + 2\mu\right)g/k - 4\left(R/\rho^o k\right)^2}{12\mu_1^2\left(s+2\mu_2\right)^2},$$

$$\delta = \frac{R/\rho^o k}{54\mu_1^3\left(s+2\mu_2\right)^3}\left[4\left(R/\rho^o k\right)^2 + 9\mu_1\left(s+2\mu_2\right)\left(s\mu_1 - \mu + 3\mu_1\mu_2\right)g/k\right],$$

Solution to problems of the second approximation (4) can be written in the following way

$$v_{ix2} = 0,\quad v_{iz2} = A^2 L_i e^{2z-2bt},\quad i = 1,2,\quad p_2 = A^2 e^{-2bt}\left[L_3\left(e^{2z}-1\right) - \frac{1}{2v_0^2}\right],$$

$$\gamma_2 = A^2 L_4 e^{2z-2bt},\quad \xi_2 = \frac{A^2}{2v_o^4}e^{-2bt}\cos 2\chi,$$

where

$$L_1 = -\frac{\alpha_o}{\left(1-\alpha_o\right)}L_2,\quad L_2 = \frac{\left(1-\alpha_o\right)H}{2\left(r - b\left(s\mu_1 + 2\mu\right)\right)},\quad L_4 = \frac{\left(1-\alpha_o\right)H}{2b\left(r - b\left(s\mu_1 + 2\mu\right)\right)},$$

$$L_3 = \frac{-\left(1-\alpha_o\right)\left(\mu_1 r - b\mu_1\left(s+2\mu_2\right)\right)\left(M_1^2 + N_1^2\right) - \alpha_o\left(\mu_2 r - b\mu_1\left(s+2\mu_2\right)\right)\left(M_2^2 + N_2^2\right)}{2\left(r - b\left(s\mu_1 + 2\mu\right)\right)},$$

$$H = \mu_1\left(s+2\right)\left(M_1^2 + N_1^2\right) - \left(s\mu_1 + 2\mu_2\right)\left(M_2^2 + N_2^2\right).$$

Substituting expressions of the first and second approximations in the equations and boundary conditions of the problem (5), we obtain the problem for the third approximation in an explicit form

$$-\alpha_o \frac{\partial \gamma_3}{\partial t} + (1-\alpha_o)\nabla\cdot \mathrm{v}_{13} = 2\alpha_0 A^3 e^{3z-3bt} L_4 \left(M_1 \sin\chi - N_1 \cos\chi\right), \qquad (6)$$

$$\frac{\partial \gamma_3}{\partial t} + \nabla\cdot \mathrm{v}_{23} = 2A^3 e^{3z-3bt} L_4 \left(N_2 \cos\chi - M_2 \sin\chi\right),$$

$$\left(\mu_1 + \frac{s}{2}\mu_1\alpha_o\right)\frac{\partial \mathrm{v}_{1x3}}{\partial t} - \frac{s}{2}\mu_1\alpha_o \frac{\partial \mathrm{v}_{2x3}}{\partial t} - r\alpha_o\left(\mathrm{v}_{2x3} - \mathrm{v}_{1x3}\right) + \frac{\partial p_3}{\partial x} = Ae^{z-bt}/2\Big(\big\{\mu_1\big[(2+s\alpha_0)\cdot$$

$$\cdot(N_1 + bM_1) - s\alpha_0 (N_2 + bM_2)\big](ta_2)' + A^2 e^{2z-2bt}\big[s\mu_1\alpha_0 L_4 (N_1 - N_2) - 2\alpha_0 r L_4 \cdot$$

$$\cdot(M_1 - M_2) - \mu_1 (2+s) L_1 M_1\big]\big\}\cos\chi + \big\{\mu_1\big[s\alpha_0 (M_2 - bN_2) - (2 + s\alpha_0)(M_1 - bN_1)\big]\cdot$$

$$\cdot(ta_2)' - A^2 e^{2z-2bt}\big[\mu_1 (2+s) L_1 N_1 + s\mu_1\alpha_0 L_4 (M_1 - M_2) + 2\alpha_0 r L_4 (N_1 - N_2)\big]\Big\}\sin\chi\Big),$$

$$\left(\mu_1 + \frac{s}{2}\mu_1\alpha_o\right)\frac{\partial \mathrm{v}_{1z3}}{\partial t} - \frac{s}{2}\mu_1\alpha_o \frac{\partial \mathrm{v}_{2z3}}{\partial t} - r\alpha_o\left(\mathrm{v}_{2z3} - \mathrm{v}_{1z3}\right) + \frac{\partial p_3}{\partial z} = Ae^{z-bt}/2\Big(\big\{\mu_1\big[(2+s\alpha_0)\cdot$$

$$\cdot(M_1 - bN_1) - s\alpha_0 (M_2 - bN_2)\big](ta_2)' + A^2 e^{2z-2bt}\big[s\mu_1\alpha_0 L_4 (M_1 - M_2) + 2\alpha_0 r L_4 \cdot$$

$$\cdot(N_1 - N_2) + \mu_1 (6 + s + 2s\alpha_0) L_1 N_1 - 2s\mu_1\alpha_0 L_2 N_2\big]\big\}\cos\chi + \big\{\mu_1\big[(2+s\alpha_0)\cdot$$

$$\cdot(N_1 + bM_1) - s\alpha_0 (N_2 + bM_2)\big](ta_2)' + A^2 e^{2z-2bt}\big[s\mu_1\alpha_0 L_4 (N_1 - N_2) - 2\alpha_0 r L_4 \cdot$$

$$\cdot(M_1 - M_2) - \mu_1 (6 + s + 2s\alpha_0) L_1 M_1 + 2s\mu_1\alpha_0 L_2 M_2\big]\big\}\sin\chi\Big)$$

$$\left(\mu_2 + \frac{s}{2}\mu_1(1-\alpha_o)\right)\frac{\partial \mathrm{v}_{2x3}}{\partial t} - \frac{s}{2}\mu_1(1-\alpha_o)\frac{\partial \mathrm{v}_{1x3}}{\partial t} + \frac{\partial p_3}{\partial x} + r(1-\alpha_o)(\mathrm{u}_{23} - \mathrm{u}_{13}) =$$

$$= Ae^{z-bt}/2\Big(\Big\{2\mu_2 (N_2 + bM_2) - s\mu_1 (1-\alpha_0)(N_1 + bM_1 - N_2 - bM_2)\big](ta_2)' +$$

$$+A^2 e^{2z-2bt}\big[s\mu_1\alpha_0 L_4 (N_1 - N_2) - (2\mu_2 + s\mu_1) L_2 M_2 - 2\alpha_0 r L_4 (M_1 - M_2)\big]\big\}\cos\chi +$$

$$+\Big\{s\mu_1 (1-\alpha_0)(M_1 - bN_1 - M_2 + bM_2) + 2\mu_2 (bN_2 - M_2)\big](ta_2)' +$$

$$-A^2 e^{2z-2b}\big[s\mu_1\alpha_0 L_4 (M_1 - M_2) + (2\mu_2 + s\mu_1) L_2 N_2 + 2\alpha_0 r L_4 (N_1 - N_2)\big]\big\}\sin\chi\Big)$$

$$\left(\mu_2 + \frac{s}{2}\mu_1(1-\alpha_o)\right)\frac{\partial \mathrm{v}_{2z3}}{\partial t} - \frac{s}{2}\mu_1(1-\alpha_o)\frac{\partial \mathrm{v}_{1z3}}{\partial t} + \frac{\partial p_3}{\partial z} + r(1-\alpha_o)(\mathrm{v}_{23} - \mathrm{v}_{13}) =$$

$$= A^3 e^{z-bt}/2\Big(\Big\{\big[2\mu_2(M_2 - bN_2) - s\mu_1(1-\alpha_0)(M_1 - bN_1 - M_2 + bN_2)\big](ta_2)' +$$

$$+ A^2 e^{2z-2bt}\big[s\mu_1\alpha_0 L_4(M_1 - M_2) + (6\mu_2 + 3s\mu_1 - 2s\alpha_0\mu_1)L_2 N_2 + 2\alpha_0 r L_4(N_1 - N_2) -$$

$$-2s\mu_1(1-\alpha_0)L_1 N_1\big]\Big\}\cos\chi + \Big\{\big[2\mu_2(N_2 + bM_2) - s\mu_1(1-\alpha_0)(N_1 + bM_2 -$$

$$-N_2 - bN_2)\big](ta_2)' + A^2 e^{2z-2bt}\big[2s\mu_1(1-\alpha_0)L_1 M_1 - 2\alpha_0 r L_4(M_1 - M_2) + s\mu_1\alpha_0 L_4 \cdot$$

$$\cdot(N_1 - N_2) - (6\mu_2 + 3s\mu_1 - 2s\alpha_0\mu_1)L_2 M_2\big]\Big\}\sin\chi\Big)$$

$$\frac{\partial\xi_3}{\partial t} - (1-\alpha_o)v_{1z3} - \alpha_o v_{2z3} = \frac{A^3 e^{-3bt}}{8v_0^6}\Big(9e^{-2bt}\left[\sin 3\chi - b\cos 3\chi\right] + \Big\{8bv_0^4 A^{-2} e^{2bt}\cdot$$

$$\cdot(ta_2)' + 16v_0^4\big[\alpha_0 L_2 + (1-\alpha_0)L_1 - v_0^2\alpha_0 L_4 d^{-1}(\mu_1 - \mu_2)(2r - b\mu_1(s + 2\mu_2))\big] + b\Big\}$$

$$\cdot\cos\chi + \Big\{-8v_0^4 A^{-2} e^{2bt}(ta_2)' + 16v_0^6\alpha_0\mu_1 L_4 d^{-1}(s + 2\mu_2)(\mu_1 - \mu_2)\Big\}\sin\chi\Big), \quad z = 0,$$

$$p_3 - v_o^2\xi_3 = -\frac{A^3 e^{-3bt}}{8v_0^4}\Big\{3\cos 3\chi + \big[5 + 16v_0^2 L_3 + 8\alpha_0 v_0^4 L_4(\mu_1 - \mu_2)\big]\cos\chi\Big\}, \quad z = 0.$$

To solve the problem (6), we set amplitude ratios functions of time to be determined. Then to an accuracy of the third approximation, solution to the nonlinear problem has the form of

$$c = 1 + \varepsilon^2 A^2 \frac{1 - e^{-2bt}}{bt} C_a, \tag{7}$$

$$v_{1x} = Ae^{z-bt}(M_1\cos\chi + N_1\sin\chi) + \varepsilon^2 A^3 e^{z-bt}\Big\{e^{2z-2bt}Q_1\sin\chi + \big[D_{v3}(e^{-2bt} - 1) + e^{2z-2bt}R_1\big]\Big\}\cos\chi,$$

$$v_{2x} = Ae^{z-bt}(M_2\cos\chi + N_2\sin\chi) + \varepsilon^2 A^3 e^{z-bt}\big[(e^{2z-2bt}Q_2 - C_{w4} - D_{w4}e^{-2bt})\sin\chi +$$

$$+(C_{v4} + D_{v4}e^{-2bt} + e^{2z-2bt}R_2)\cos\chi\big],$$

$$v_{1z} = Ae^{z-bt}(M_1\sin\chi - N_1\cos\chi) + \varepsilon A^2 L_1 e^{2z-2bt} + \varepsilon^2 A^3 e^{z-bt}\big[(D_{v3}(e^{-2bt} - 1) + e^{2z-2bt}Q_3)\sin\chi +$$

$$+e^{2z-2bt}R_3\cos\chi\big],$$

$$v_{2z} = Ae^{z-bt}(M_2\sin\chi - N_2\cos\chi) + \varepsilon A^2 L_2 e^{2z-2bt} + \varepsilon^2 A^3 e^{z-bt}\big[(C_{v4} + D_{v4}e^{-2bt} + e^{2z-2bt}Q_4)\sin\chi +$$

$$+(C_{w4} + D_{w4}e^{-2bt} + e^{2z-2bt}R_4)\cos\chi\big],$$

$$p = Ae^{z-bt}\cos\chi + \varepsilon A^2 e^{-2bt}\left[L_3(e^{2z} - 1) - \frac{1}{2v_0^2}\right] + \varepsilon^2 A^3 e^{z-bt}\big[(C_{v5} + D_{v5}e^{-2bt} + e^{2z-2bt}Q_5)\sin\chi +$$

$$+(C_{w5} + D_{w5}e^{-2bt} + e^{2z-2bt}R_5)\cos\chi\big],$$

$$\gamma = \varepsilon A^2 L_4 e^{2z-2bt} + \varepsilon^2 A^3 e^{3z-3bt}\left[Q_6 \sin\chi + R_6 \cos\chi\right],$$

$$\xi = \frac{A}{\nu_o^2} e^{-bt}\cos\chi + \varepsilon \frac{A^2}{2\nu_o^4} e^{-2bt}\cos 2\chi + \varepsilon^2 e^{-bt} A^3\left[3e^{-2bt}A^3 8^{-1}\nu_0^{-6}\cos 3\chi + \right.$$

$$+\Big(\left(1+9b^2\right)^{-1} e^{-2bt}\Big\{2\alpha_0 d^{-1} L_4\left(\mu_1-\mu_2\right)\left[6br+\left(\mu_1 s+2\mu_2\right)\left(1-3b^2\right)\right]-2\nu_0^{-2}\cdot$$

$$\cdot\left(1+3b^2\right)C_a+\left(1-\alpha_0\right)\left(Q_3-3bR_3+D_{v3}\right)+\alpha_0\left(Q_4-3bR_4+D_{v4}-3bD_{w4}\right)+8^{-1}\nu_0^{-6}\cdot$$

$$\cdot\left(5-3b^2\right)\Big\}+\left(1+b^2\right)^{-1}\left[\left(1-\alpha_0\right)C_{v3}+\alpha_0\left(C_{v4}-bC_{w4}\right)\right]\Big)\cos\chi+\Big(\left(1+9b^2\right)^{-1} e^{-2bt}\cdot$$

$$\cdot\Big\{4\alpha_0 d^{-1} L_4\left(\mu_1-\mu_2\right)\left[r-2b\left(\mu_1 s+2\mu_2\right)\right]+4\nu_0^{-2} bC_a-\left(1-\alpha_0\right)\left(R_3+3bQ_3+\right.$$

$$\left.+3bD_{v3}\right)-\alpha_0\left(R_4+3bQ_4+D_{w4}+3bD_{v4}\right)+2\nu_0^{-6} b\Big\}-\left(1+b^2\right)^{-1}\left[\left(1-\alpha_0\right)bC_{v3}+\right.$$

$$\left.\left.+\alpha_0\left(C_{w4}+bC_{v4}\right)\right]\Big)\sin\chi\right].$$

Here coefficients of the third approximation are omitted because of their awkwardness.

(7) implies that over time, frequency (phase velocity) tends to the value corresponding to the linear problem. Fig.1 shows the dependence of the wave phase velocity c^* on time t^* when $\rho_2{}^0 = 1500$ kg/m³ and $\rho_2{}^0 = 500$ kg/m³ ($A = 1$, $\lambda = 5$m, $\rho_1{}^0 =$ 1000 kg/m³, $\eta = 10^{-3}$ kg/(m*c), a = 0,15 * 10^{-2} m). Phase velocity is greater when particles of disperse phase have greater density than the carrier phase which is not the case with lighter particles. From graphs showing dependence of wave amplitude on time on Fig. 2 it is seen that the wave motion decays faster than phase velocity reaches a linear value.

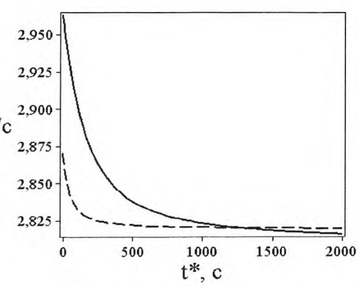

Fig. 1. Dependence t^*: — $\rho_2{}^0$=500кг/м³, — $\rho_2{}^0$=1500кг/м³.

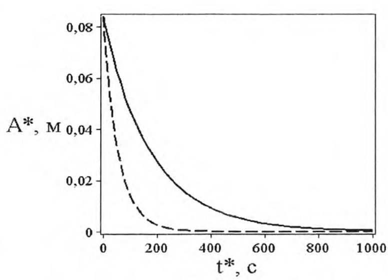

Fig. 2. Dependence of amplitude on $c^*(t^*)$: — $\rho_2{}^0$= 500 кг/м³, — $\rho_2{}^0$= 1500 кг/м³

Dimensionless coordinates of disperse and carrier particles satisfy the equations

$$x_i = \chi_i + \gamma_i t_i - \theta,\quad \chi_{it} = \varepsilon\left(\frac{t_i\gamma_i}{c}\right)' \mathrm{v}_{xi} - \left(t_i\gamma_i\right)',\quad z_{it} = \varepsilon\left(\frac{t_i\gamma_i}{c}\right)' \mathrm{v}_{zi}, \tag{8}$$

where $\gamma_i(t) = \omega/\sigma_i$ is vibration frequency of i-phase particle.

Solution (8) is found in expanded form in small parameter

$$\chi_i = \sum_{j=0}^{\infty} \varepsilon^j \chi_{ij}, \quad z = \sum_{j=0}^{\infty} \varepsilon^j z_{ij}, \tag{9}$$

$$\gamma_i = 1 + \sum_{j=0}^{\infty} \varepsilon^j \gamma_{ij}.$$

Substituting expressions (7), series (9) in the motion equation (8), to determine the first three coefficients we obtain the following equations:

$$\frac{d\chi_{i0}}{dt_i} = -1, \quad \frac{dz_{i0}}{dt_i} = 0,$$

$$\frac{d\chi_{i1}}{dt_i} = Ae^{z_{i0}-bt_i}\left[M_i \cos\chi_0 + N_i \sin\chi_0\right] - \left(t_i\gamma_1\right)',$$

$$\frac{dz_{i1}}{dt_i} = Ae^{z_{i0}-bt_i}\left[M_i \sin\chi_{i0} - N_i \cos\chi_{i0}\right], \quad \chi_{i0} = x_{i0} + d,$$

$$\frac{d\chi_{i2}}{dt_i} = Ae^{z_{i0}-bt_i}\left\{\left[N_i x_{i1} + M_i z_{i1} + M_i\left(t_i\gamma_{i1}\right)' - bM_i t_i \gamma_{i1}\right]\cos\chi_{i0} + \right.$$

$$\left. + \left[N_i z_{i1} - M_i x_{i1} + N_i\left(t_i\gamma_{i1}\right)' - bN_i t_i \gamma_{i1}\right]\sin\chi_{i0} - \left(t_i\gamma_{i2}\right)'\right\},$$

$$\frac{dz_{i2}}{dt_i} = Ae^{z_{i0}-bt_i}\left\{\left[N_i x_{i1} + M_i z_{i1} + M_i\left(t_i\gamma_{i1}\right)' - bM_i t_i \gamma_{i1}\right]\sin\chi_{i0} - \right.$$

$$\left. - \left[N_i z_{i1} - M_i x_{i1} + N_i\left(t_i\gamma_{i1}\right)' - bN_i t_i \gamma_{i1}\right]\cos\chi_{i0} + Ae^{z_0-bt_i} L_i\right\},$$

First, we have $\chi_{i0} = \chi_{iL} - t_i = x_{iL} - t_i + \theta, z_{i0} = z_{iL}$, where x_{iL}, z_{iL} are Lagrangian coordinates of the particle at a standstill.

Determining the periodic solution for χ_{i1}, z_{i1}, we obtain

$$\gamma_{i1} = 0, \quad \chi_{i1} = \frac{Ae^{z_{iL}-bt_i}}{b^2+1}\left[\left(N_i - bM_i\right)\cos\chi_{iL} - \left(M_i + bN_i\right)\sin\chi_{iL}\right],$$

$$z_{i1} = \frac{Ae^{z_{iL}-bt_i}}{b^2+1}\left[\left(M_i + bN_i\right)\cos\chi_{iL} + \left(N_i - bM_i\right)\sin\chi_{iL}\right],$$

Equations for determining the trajectory of a fluid particle in the second approximation take the form

$$\frac{dx_{i2}}{dt_i} = \frac{A^2 e^{2z_{iL}-2bt_i}}{b^2+1}\left(M_i^2 + N_i^2\right) - \left(t_i\gamma_{i2}\right)', \quad \frac{dz_{i2}}{dt_i} = A^2 e^{2z_{iL}-2bt_i}\left(L_i - \frac{bM_i^2 + bN_i^2}{b^2+1}\right).$$

Solving these equations, we obtain

$$x_{i2}=0,\quad z_{i2}=\frac{A^2e^{2z_{iL}}}{2b}\left(1-e^{-2bt_i}\right)\left(L_i-\frac{bM_i^2+bN_i^2}{b^2+1}\right),$$

$$\gamma_{i2}=\frac{A^2e^{2z_{iL}}}{2bt_i\left(b^2+1\right)}\left(1-e^{-2bt_i}\right)\left(M_i^2+N_i^2\right).$$

The value $u_{si}=c-\sigma_i/k$ is transfer velocity. Its approximation is of the form

$$u_{si}=\varepsilon^2c_0\frac{A^2e^{2z_{iL}}}{2bt_i\left(b^2+1\right)}\left(1-e^{-2bt_i}\right)\left(M_i^2+N_i^2\right).$$

Transfer velocity depends not only upon the depth the particle is located at, but also on attenuation time. Particle motion consists of two convergent modes—nonperiodical and rotational. Figs. 3, 4 show trajectories of disperse phase particles at ρ_2^0 = 500 kg/m³ and ρ_2^0 = 1500 kg/m³.

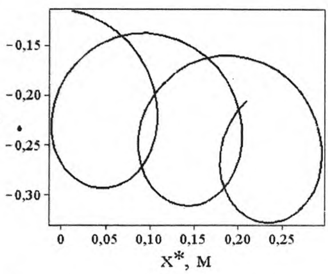

Fig. 3. Trajectories of disperse phase particles at ρ_2^0 = 500 kg/m³

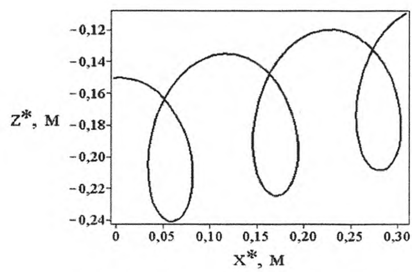

Fig. 4. Trajectories of disperse phase particles at ρ_2^0 = 1500 kg/m³

The graphs show that the lighter particles deepen while moving, and the heavier, on the contrary, rise closer to the free surface. That leads to an increase (decrease) in concentration of heavier (lighter) particles in the near-surface layer. The present paper analytically discovered this phenomenon of nonlinear character [3]. A more precise solution obtained in this work confirms and specifies this nonlinear effect.

REFERENCES

1. Aleshkov, Ju.Z. Diffusion of Impurities in the Ocean. *Vestnik Sankt-Peterburgskogo universiteta — Saint Petersburg University Herald.* 2001. Series 1. Issue. 1. Pp. 12-20.

2. Nudner, I.S., Maksimov, V.V., Majorov, Ju.B., Babchik, D.V. Modelling of Pump Dynamics in the Coastal Area. *Vychislitel'nye tehnologii — Computational Technologies.* 2008. Vol. 13, № S2. Pp. 60-77.

3. Barinov, V.A., Butakova, N.N. A Nonlinear Problem of Surface Waves on a Two-Phase Mixture. *Zhurnal vychislitel'noj matematiki i matematicheskoj fiziki — Computational Mathematics and Mathematical Physics.* 2003. Vol. 43, №. 12. Pp. 1870-1883.

4. Barinov, V.A., Butakova, N.N. Propagation of Waves on the Free Surface of a Two-Phase Mixture. *Izvestija RAN — Bulletin of the Russian Academy of Sciences*. 2003. № 6. Pp. 94-102.

5. Barinov, V.A., Butakova, N.N. Waves on the Free Surface of a Two-Phase Medium. *Prikladnaja mehanika i tehnicheskaja fizika — Journal of Applied Mechanics and Technical Physics*. 2002. Vol. 43, № 4. Pp. 27-35.

6. Aleshkov, Ju.Z., Barinov, V.A., Butakova, N.N. Nonlinear Surface Waves on a Layer of a Two-Phase Medium. *Vestnik Sankt-Peterburgskogo universiteta — Saint Petersburg University Herald*. 2003. Series 1. Issue 4. Pp. 64-75.

7. Sretenskij, L.N. *Teorija volnovyh dvizhenij zhidkosti* [Theory of Wave Motion in Fluid]. M.: Nauka, 1977. 816 p.

8. Aleshkov, Ju.Z. *Teorija voln na poverhnosti tjazheloj zhidkosti* [Wave Theory for Heavy Liquid Surface]. L.: Leningrad University Publ., 1981. 196 p.

9. Barinov, V.A., Basinskij, K.Ju. Nonlinear Stokes Waves on the Surface of Low-Viscosity Fluid. *Vestnik Udmurtskogo universiteta — Udmurt University Herald*. 2011. № 2. Pp. 112-122.

10. Barinov, V.A., Basinskij, K.Ju. The Solution of a Nonlinear Problem of Waves on the Surface Weakly-Viscous Fluid. *Vestnik Sankt-Peterburgskogo universiteta — Saint Petersburg University Herald*. 2011. Series 10. Issue. 2. Pp. 9-16.